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On the Applicative Phase of a Mathematical Principle

In a paper read before the Baton Rouge meeting of the National Council of Teachers of Mathematics, E. G. Olds elaborated the topic "Use of Applications for Instructional Purposes". His viewpoint suggests, even connotes, some wide, possibly profound, generalizations.* His own statement "To see how a mathematical law is applied is to gain a better understanding of its meaning", and the quoted words of President Wickenden "The student learns principles best in connection with their applications" convey a meaning strangely at variance with that of the alleged remark of a certain Professor of pure mathematics. Upon being asked by one of his students what uses could be made of the mathematics he had just finished expounding, he is said to have replied "Thank God there is not one use to which it can be applied."

This is a period of the race's history in which as never before the interests of mere individual must be held for sacrifice, if they are not also the interests of the community, or, wider still, the human group. In times when human needs cry to high heaven for relief, a worshipper before the Shrine of Truth in some wilderness remote from the distress calls of his fellows is apt to be confused when he finds that Truth has left its shrine and gone to minister to the distress calls. A science which insists on remaining entirely separate from any actual or potential human service application is scarcely worthy of its name.

To undertake seriously to apply a mathematical principle in the solution of a problem featured by real and human elements is to find the *limitations* of that principle. Knowledge of these limitations is therefore a contribution to the mastery of the principle.

This page is too restricted for complete exposition of our thesis. Let us close with an illustration of the absurdity of devotion to Truth "though the heavens fall". The illustration is the picture brought up by the story of two chess-players who were so bent upon their search for truth of chess (the issue of the game) that they knew not that the house had caught fire. So, in the same manner the furniture was removed, the players, game and table, were taken from the burning house.

S. T. Sanders.

*See *Mathematics Teacher*, February, 1941.

Some Introductory Exercises in the Manipulation of Fourier Transforms*

By ROBERT H. CAMERON
Massachusetts Institute of Technology

1. *Introduction.* In this paper we will not be so much interested in the intrinsic properties of Fourier transforms themselves as in what we can do with them. "What formal manipulations can we carry on and what problems can we solve by using the Fourier transforms as one of our tools?" will be the questions we try to answer. It might therefore be well at the outset before even telling what a Fourier transform is, to give a few samples of problems it can solve for us. Perhaps if the problems interest the reader and their formal solutions intrigue or mystify him, he will be willing to read further and find out how those queer looking solutions were obtained.

Let us take as one sample the non-linear integral equation

$$\int_{-\infty}^{\infty} f(x-t)f(t)dt + 2f(x) = g(x); \tag{1.1}$$

in which $g(x)$ is a given function of the real variable x, and $f(x)$ is the unknown function that we wish to find. To make the problem even more specific, let us assume that the given function is

$$g(x) = \frac{4x^2+10}{\pi(x^4+5x^2+4)} .$$

Now I think the reader will agree that this is a problem to which no ordinary formal methods of approach apply. Linear integral equations are bad enough, but this is not even linear since $f(x-t)$ and $f(t)$ are multiplied together. Yet it is possible by means of Fourier transforms to write down a formal answer to this problem, and then by accurate analysis justify the formal process under certain conditions. In order (we hope) to whet the reader's curiosity, we shall write out the formal solution to (1.1) immediately, withholding the explanation of how it was obtained until a later part of the paper. Here it is:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\left\{\left[1+\int_{-\infty}^{\infty} e^{iu\xi}g(\xi)d\xi\right]^{\frac{1}{2}} - 1\right\} du.$$

*This is the fifth article in a series of expository articles solicited by the Editors.

If we use the specific function

$$g(x)=\frac{4x^2+10}{\pi(x^4+5x^2+4)}$$

and substitute in the above formula, we find (after carrying out the indicated operations with the aid of a table of definite integrals) that

$$f(x)=\frac{1}{\pi(x^2+1)}\cdot$$

Moreover we can readily verify by substituting this function in the original equation (1.1) that it is actually a solution of the equation.

Perhaps the reader might be interested at the start to see a few other equations whose solutions will be found by means of Fourier transforms. In many cases the solutions have to be left in the form of definite integrals, since these integrals cannot be evaluated finitely in the terms of elementary functions. However, even being able to express the answer in such a form is better than not being able to express it at all. For instance, we shall see that the differential equation

$$\frac{d^2Y}{dx^2}+\frac{dY}{dx}+xY=0 \tag{1.2}$$

has as its general solution

$$\begin{aligned} Y=A&\int_0^\infty e^{-t^2/2}\cos\left(tx-(t^3/3)\right)dt \\ &+B\int_0^\infty \left[e^{-t^2/2}\sin\left(tx-(t^3/3)\right)-e^{(t^2/2)-(t^3/3)-tx}\right]dt; \end{aligned} \tag{1.3}$$

that the integral equation

$$\rho(x)+\int_0^\infty \rho(x-t)e^{-t}dt = \frac{1}{x^2+1} \tag{1.4}$$

has the bounded solution

$$\rho(x)=\int_0^\infty \frac{(2+u^2)\cos ux-u\sin ux}{4+u^2}\,e^{-u}du$$

and that the difference-differential equation

$$\frac{d}{dx}[f(x)]+f(x)+f(x+1)=\frac{1}{x^2+1} \tag{1.5}$$

has the bounded solution

$$f(x)=\int_0^\infty \frac{\cos xs+\cos(xs-s)+s\sin xs}{2+2\cos s+2s\sin s+s^2}\,e^{-s}ds.$$

2. *The formal definition of a Fourier transform.* The average paper on Fourier transforms or their applications is apt to present a rather forbidding aspect to the casual reader. One is assumed to have rather extensive knowledge of the Lebesgue integral and its properties; and in particular one is assumed to be very much at home in the spaces L_1, L_2, and L_p. Moreover it is usually taken for granted that the reader is well acquainted with the whole (very extensive) literature of Fourier Transforms, and that he is able to fit that particular paper right into the appropriate notch. Worst of all the theorems themselves are apt to merely deepen the mystery of the subject and completely discourage the reader; for in a large number of cases the hypotheses and conclusions seem to be entirely haphazard, having no relation whatsoever to anything else in mathematics, or even to each other. Guessing the behavior of the stock market five years in advance seems a small matter in comparison to guessing what conclusions will go with what hypotheses in the theory of Fourier transforms.

Now obviously a great deal goes on under the surface; for mathematicians do not go around making altogether haphazard guesses, and then pulling out of a hat chains of logic which prove these guesses to be correct. This underlying creative thinking is almost certain to be obscured in the mass of detailed work necessary to prove the theorems rigorously; and if the author does not give a preliminary sketch or final summary of his work, his most important ideas may be lost to the reader who is not a specialist. All this is unfortunate, because many men who might advantageously use Fourier transforms as a tool in their work are discouraged and prevented from doing so.

In order to avoid this difficulty and show the simplicity of the underlying ideas of the subject, we shall in this introductory paper lay aside all ideas of mathematical rigor. We shall not be explicit as to the type of integrals used or the sense in which infinite integrals are to be interpreted. These things are to exist in some reasonable sense; and just what is "reasonable" belongs to a later phase of the subject. In this spirit we make the following definition.

Definition: *If $f(x)$ is a given function, and $i=\sqrt{-1}$, the function*

$$(2.1)\qquad F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(s)ds$$

is called the Fourier transform of $f(x)$; and we shall denote this relationship in the following way:

$$f(x)\rightleftarrows F(x).$$

This definition will seem more concrete if we actually apply it to a particular function and find its Fourier transform. Let us apply it to

$$f(x)=\exp(-a^2x^2)=e^{-a^2x^2},$$

(where a is a positive number). Replacing x by the variable of integration s, we have $f(s)=\exp(-a^2s^2)$; and substituting in (2.1), we have

$$(2.2)\qquad F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}e^{-a^2s^2}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-(x^2/4a^2)-(as-(ix/2a))^2}ds.$$

Let $t=as-(ix)/(2a)$, and substitute in (2.2) then we have*

$$F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-(x^2/4a^2)-t^2}dt/a$$

$$=\frac{e^{-(x^2/4a^2)}}{a\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-t^2}dt=\frac{e^{-(x^2/4a^2)}}{a\sqrt{2}},$$

since the probability integral

$$\int_{-\infty}^{\infty}\exp(-t^2)dt=\sqrt{\pi}$$

(see for instance B. O. Pierce's table of integrals or Wood's *Advanced Calculus*). We have thus obtained the result:

$$e^{-a^2x^2}\rightleftarrows\frac{1}{a\sqrt{2}}\,e^{-(x^2/4a^2)},$$

*Since x goes from $-\infty$ to $+\infty$ along the real axis, t really goes from

$$-\infty-(ix)/(2a)\quad\text{to}\quad+\infty-(ix)/(2a)$$

along a line parallel to the real axis and $x/(2a)$ units below it. But it is easy to see that this line can be moved up to the real axis without changing the value of the integral.

and in particular, when $a = 1/\sqrt{2}$,

$$e^{-\frac{1}{2}x^2} \rightrightarrows e^{-\frac{1}{2}x^2}.$$

Thus we see that $\exp(-\frac{1}{2}x^2)$ is its own Fourier transform; and we anticipate that it will play an important role in the theory for this reason.

3. *Formal properties of Fourier transforms.* The concept we have just defined is a useful one because of its many useful formal properties. Some of these are:

(a) *It is linear.* This means that if

$$f(x) \rightrightarrows F(x) \quad \text{and} \quad g(x) \rightrightarrows G(x),$$

then

$$f(x)+g(x) \rightrightarrows F(x)+G(x)$$

and

$$cf(x) \rightrightarrows cF(x)$$

(where c is any real or complex constant). These facts are self evident when we write them out:

$$\int_{-\infty}^{\infty} e^{isx}[f(s)+g(s)]\,ds = \int_{-\infty}^{\infty} e^{isx}f(s)\,ds + \int_{-\infty}^{\infty} e^{isx}g(s)\,ds$$

$$\int_{-\infty}^{\infty} e^{isx}[cf(s)]\,ds = c\int_{-\infty}^{\infty} e^{isx}f(s)\,ds.$$

(b) *It replaces multiplication by ix by differentiation.* Symbolically

$$ixf(x) \rightrightarrows \frac{d}{dx}F(x);$$

or written out,

$$\frac{d}{dx}\int_{-\infty}^{\infty} e^{isx}f(s)\,ds = \int_{-\infty}^{\infty} e^{isx}[isf(s)]\,ds.$$

This formula can be verified by merely carrying out the indicated differentiation; and it holds whenever differentiation under the integral sign is permissable. We shall of course not worry about such a detail at present, but will operate with this formula as though it were universally true, and check up after all formal operations are completed.

(c) *It replaces differentiation by multiplication by* $-ix$. This property is practically the same as the preceding, except that the differentiation is applied to the original function and the multiplication is applied to the transform. Symbolically

$$\frac{d}{dx}f(x) \rightleftarrows -ix\,F(x)\ ; \tag{3.1}$$

or written out

$$-ix\int_{-\infty}^{\infty} e^{isx}f(s)ds = \int_{-\infty}^{\infty} e^{isx}\left[\frac{d}{ds}f(s)\right] ds. \tag{3.2}$$

To see this formal relationship, integrate the right hand member by parts. We have

$$\int_{-\infty}^{\infty} e^{isx}f'(s)ds = \Big[(e^{isx})f(s)\Big]_{s=-\infty}^{s=+\infty} - \int_{-\infty}^{\infty} ixe^{isx}f(s)ds. \tag{3.3}$$

Now if $f(s)\to 0$ as $s\to\pm\infty$, the expression in brackets drops out, and the integral which remains equals the left member of (3.2). The condition that $f(s)\to 0$ as $s\to\pm\infty$ will usually hold for the functions with which we deal; and we will regard (3.1) as a formal identity for practical manipulative purposes.

4. *A second order differential equation.* Before going any further with our study of the properties of Fourier transforms, we shall see how the second example mentioned in the introduction can be partially solved by the use of properties (a), (b), (c) alone. Let us suppose that the solution $Y(x)$ of the differential equation

$$\frac{d^2Y}{dx^2}+\frac{dY}{dx}+xY=0$$

is the Fourier transform of a function $y(x)$; and let us see what differential equation $y(x)$ must satisfy. Then since $y(x)\rightleftarrows Y(x)$, we have by (b):

$$ix\,y(x)\rightleftarrows\frac{d}{dx}\,Y(x),$$

and

$$(ix)^2y(x)\rightleftarrows\frac{d^2}{dx^2}\,Y(x).$$

Also by (c)
$$\frac{d}{dx}\, y(x) \rightleftarrows -ix\; Y(x),$$

or
$$\frac{1}{-i}\;\frac{d}{dx}\, y(x) \rightleftarrows xY(x);$$

so that we obtain finally by using (a),

$$-x^2y(x)+ix\,y(x)+i\,\frac{d}{dx}\,y(x) \rightleftarrows \frac{d^2Y}{dx^2}+\frac{dY}{dx}+xY.$$

But the transform of zero is zero, so we expect $y(x)$ to satisfy the equation

$$-x^2y+ixy+i\,\frac{dy}{dx}=0$$

or
$$\frac{dy}{dx}=-(ix^2+x)y.$$

Moreover this equation is of the first order and the variables are separable, so we may write

$$\int \frac{dy}{y}=-\int (ix^2+x)dx$$

and
$$\log y=-i\,\frac{x^3}{3}+\frac{x^2}{2}+\log c$$

and
$$y(x)=ce^{-i(x^3/3)-(x^2/2)}.$$

Thus we have solved the transformed equation and found $y(x)$; and since $y(x) \rightleftarrows Y(x)$, we have

$$Y(x)=\frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}e^{-i(s^3/3)-(s^2/2)}ds.$$

We can express this answer in terms of real quantities by using the fact that $\exp(iu)=\cos u+i\sin u$, and we obtain on putting

$$A=c\sqrt{2/\pi}\;,$$

$$Y(x)=\frac{A}{2}\int_{-\infty}^{\infty} e^{i(sx-(s^3/3))}e^{-(s^2/2)}ds$$

$$= \frac{A}{2}\int_{-\infty}^{\infty} \cos(sx-(s^3/3))e^{-(s^2/2)}ds$$

$$+i\frac{A}{2}\int_{-\infty}^{\infty} \sin(sx-(s^3/3))e^{-(s^2/2)}ds$$

$$(4.1) \qquad = A\int_{0}^{\infty} \cos(sx-(s^3/3))e^{-(s^2/2)}ds.$$

In the last step, the sine integral vanishes since its positive and negative parts cancel; and the cosine integral from $-\infty$ to $+\infty$ is twice its value from 0 to $+\infty$ since the cosine is an even function.

The reader will note that we have obtained only one part of the solution (1.3) and may wonder why. The answer is that the second part is not the Fourier transform of any function; so when we assumed that $Y(x)$ was the transform of $y(x)$, we ruled out the second part. The missing part can be obtained by modifying our definition of a Fourier transform and will be discussed later in section 12. If the coefficient of dY/dx in (1.2) had been negative (say -1) instead of positive, we would have formally obtained

$$Y(x) = A\int_{0}^{\infty} \cos(sx-(s^3/3))e^{s^2/2}ds;$$

but the exponential now becomes infinite as $s \to \pm\infty$ and the integral diverges. Thus in this case we would get neither part of the solution by the present unmodified method. However, the modification of the method given in section 12 would give both solutions in this case.

It still remains to verify that (4.1) is actually a solution of (1.2), for we have just seen that the solution need not be a Fourier transform at all. We obtained the answer by purely formal manipulation of the properties (a), (b), (c); and we have already noted that these properties depend on certain extra conditions which are not necessarily satisfied in every case. Thus we must verify first that the integral (4.1) converges and second that it satisfies the equation (1.2). We therefore note that the factor $\exp(-s^2/2)$ goes to zero so rapidly as $s \to \infty$ that the integral converges and permits all necessary manipulations; and by direct substitution in (1.2) we find that it satisfies the equation.

5. *A non-homogeneous differential equation.* If we analyze the methods used in partially solving (1.2), we find that the steps are these:

(1) We let $y(x) \rightleftarrows Y(x)$ and see what equation $y(x)$ must satisfy when $Y(x)$ satisfies a given equation. We call this equation for $y(x)$ the transformed equation.

(2) We solve the transformed equation for $y(x)$.

(3) We calculate $Y(x)$ by taking the transform of $y(x)$.

Any one of these steps may be impossible to carry out; but consider the second particularly. If the transformed equation is no simpler than the original equation, the method is useless. Now since (ix) factors go into derivatives and vice versa, the order of the new equation must equal the highest degree of the coefficients of the original equation and vice versa; so the method improves the situation only when the given differential equation has polynomial coefficients of lower degree than the order of the equation. But first order linear differential equations are the only ones we can formally solve for coefficients which are general functions of x, and it thus appears that our method is likely to be useful only when the coefficients of the given equation are linear functions of x. However, such equations

$$(5.1) \quad (a_0+b_0x)\frac{d^nY}{dx^n}+(a_1+b_1x)\frac{d^{n-1}Y}{dx^{n-1}}+\cdots+(a_n+b_nx)y=0$$

form an important class; and our method does formally apply to this type of equation.

We might next enquire whether we could solve (5.1) if the right hand side were a function of x instead of zero. For instance, apply the method to

$$\frac{d^3Y}{dx^3}-3xY=x^3e^{-\frac{1}{2}x^2}.$$

Letting $y(x) \rightleftarrows Y(x)$, we have

$$(ix)^3y(x) \rightleftarrows \frac{d^3}{dx^3}\ Y(x)$$

$$i\frac{d}{dx}\,y(x) \rightleftarrows x\ Y(x) \quad \text{and}$$

$$(5.2) \qquad -ix^3y(x)-3i\,\frac{d}{dx}\,y(x) \rightleftarrows \frac{d^3Y(x)}{dx^3}\ -3x\ Y(x).$$

But the right hand side equals $x^3 \exp(-\frac{1}{2}x^2)$, so we must find what function has this as its transform in order to know what the left side equals. Thus we see that it is necessary in working with Fourier transforms to be able to work backwards and forwards. We must not only know how to get the transform of a function, but also how to find the function corresponding to a given transform.

In the present case, this causes no difficulty, for we know that

$$e^{-\frac{1}{2}x^2} \rightleftarrows e^{-\frac{1}{2}x^2},$$

so that

$$\frac{d^3}{dx^3}\, e^{-\frac{1}{2}x^2} \rightleftarrows (-ix)^3 e^{-\frac{1}{2}x^2}\ ; \quad \text{or}$$

(5.3) $$i^3(-x^3+3x)e^{-\frac{1}{2}x^2} \rightleftarrows x^3 e^{-\frac{1}{2}x^2}.$$

Since the right members of (5.2) and (5.3) are equal, we shall assume that the left members are also equal (the validity of such an assumption will be discussed later). Thus we have

(5.4) $$-ix^3 y(x) - 3i\frac{d}{dx}\, y(x) = i(x^3-3x)e^{-\frac{1}{2}x^2}, \quad \text{or}$$

(5.5) $$\frac{d}{dx}\, y(x) + \frac{x^3}{3}\, y(x) = -\tfrac{1}{3}(x^3-3x)e^{-\frac{1}{2}x^2}.$$

Being a first order linear differential equation, this has the integrating factor

$$e^{\int (x^3/3)dx} = e^{x^4/12};$$

and multiplying (5.5) by this, we have

$$e^{x^4/12}\left\{\frac{dy}{dx} + \frac{x^3}{3}y\right\} = -\tfrac{1}{3}(x^3-3x)e^{(x^4/12)-(x^2/2)},$$

or

$$\frac{d}{dx}[ye^{x^4/12}] = -\frac{d}{dx}[e^{(x^4/12)-(x^2/2)}].$$

Integrating, we have

$$ye^{x^4/12} = -e^{(x^4/12)-(x^2/2)} + c\ ;$$

or

$$y(x) = -e^{-\frac{1}{2}x^2} + ce^{-(x^4/12)}.$$

Finally, since $y(x) \rightleftarrows Y(x)$, we obtain

$$Y(x) = -e^{-\frac{1}{2}x^2} + \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}e^{-(s^4/12)}ds.$$

The complex integral can be reduced to real form as in the preceding problem; and other parts of the solution involving other arbitrary constants can be found by the method of section 12.

6. *Fourier's theorem.* We have seen in the last problem that it is likely to be necessary not only to calculate Fourier transforms, but also their inverses. We need to know how to find the original function when its Fourier transform is given. Fortunately, this problem has a very simple formal solution; though the underlying theory is far from simple. This leads us to our fourth property of the Fourier transformation:

(d) *When repeated, it reproduces the original function with the sign of the independent variable changed.* Stated symbolically, this says that if

$$f(x) \rightrightarrows F(x),$$

then

$$F(x) \rightrightarrows f(-x);$$

or

$$f(x) \rightrightarrows F(x) \rightrightarrows f(-x).$$

Written out, this says that if

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(s) ds = F(x),$$

then (formally)

$$(6.1) \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s) ds = f(x).$$

Combining the two integrals and replacing x by s and s by t in the first, the statement is that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \left[\int_{-\infty}^{\infty} e^{its} f(t) dt \right] ds = f(x)$$

or

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{is(t-x)} f(t) dt ds = f(x).$$

This can be (and originally was) stated in terms of real numbers. Thus, if we put $\exp(iu) = \cos u + i \sin u$, we find that sine cancels out by symmetry while the cosine doubles up, and we have

$$\frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} \cos[s(t-x)] f(t) dt ds = f(x).$$

This is Fourier's theorem, which holds for a wide class of functions, though not by any means for all functions. We shall not however try to prove it in this form; but shall go back to Fourier transforms and our symbolic notation.

If we wish to indicate that $f(x) \rightleftarrows F(x)$ and $F(x) \rightleftarrows \mathfrak{F}(x)$, we will write

$$f(x) \rightleftarrows F(x) \rightleftarrows \mathfrak{F}(x).$$

However, if we are not interested in $F(x)$ and merely wish to indicate that $\mathfrak{F}(x)$ is the double transform of $f(x)$, we shall omit the $F(x)$ and merely write

$$f(x) \rightleftarrows\rightleftarrows \mathfrak{F}(x).$$

Thus property (d) says that

$$f(x) \rightleftarrows\rightleftarrows f(-x);$$

and we shall begin by verifying this for some simple functions.

We have already found that

$$e^{-\frac{1}{2}x^2} \rightleftarrows e^{-\frac{1}{2}x^2} ;$$

and of course if we apply the transformation again, we still get the same function; so

$$e^{-\frac{1}{2}x^2} \rightleftarrows\rightleftarrows e^{-\frac{1}{2}x^2} .$$

But $\exp(-\frac{1}{2}x^2) = \exp[-\frac{1}{2}(-x)^2]$, so $\exp(-\frac{1}{2}x^2)$ has the specified property (d). Let us also verify that $x^n \exp(-\frac{1}{2}x^2)$ has this property when n is a positive integer. Applying (b) n times, we have

$$(ix)^n e^{-\frac{1}{2}x^2} \rightleftarrows \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}) ,$$

and applying (c) n times, we obtain

$$\frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}) \rightleftarrows (-ix)^n e^{-\frac{1}{2}x^2} .$$

Thus

$$(ix)^n e^{-\frac{1}{2}x^2} \rightleftarrows\rightleftarrows (-ix)^n e^{-\frac{1}{2}x^2};$$

or

$$x^n e^{-\frac{1}{2}x^2} \rightleftarrows\rightleftarrows (-x)^n e^{-\frac{1}{2}(-x)^2} ;$$

and $x^n \exp(-\frac{1}{2}x^2)$ has property (d) when n is a positive integer. But since the Fourier transformation is linear, sums of functions of this type must also have the property (d) and it follows that if $P(x)$ is any polynomial, the product

$$P(x)e^{-\frac{1}{2}x^2} = (a_0x^n + a_1x^{n-1} + \cdots + a_n)e^{-\frac{1}{2}x^2} \tag{6.2}$$

has the same property. By approximating other functions by functions of the form (6.2), it is possible to show that a large class of these other functions have property (d). In his proof of Plancherel's theorem, Wiener applies limiting processes to sequences of functions of the form (6.2) and thus shows that (d) holds for the important and extensive class of functions known as *the class* L_2. A function $f(x)$ is said to belong to the class L_2 if $f(x)$ is Lebesgue integrable between every pair of finite limits a and b and $[f(x)]^2$ is absolutely integrable from $-\infty$ to $+\infty$. Of course this includes all functions $f(x)$ which are Riemann integrable on all finite intervals and for which

$$\int_{-\infty}^{+\infty} [f(x)]^2 dx$$

converges absolutely. In particular, it includes all continuous functions $f(x)$ which approach zero at $\pm\infty$ as fast or faster than $1/x$ does. Thus the functions

$$(6.3) \qquad \frac{1}{\sqrt{x^2+1}}, \quad e^{-x^4}, \qquad \frac{\sin x}{x^2+1}, \quad \text{etc.}$$

belong to L_2 and so have property (d). Another class of functions having the property (d) is the class L_1 which consists of functions that are absolutely integrable from $-\infty$ to $+\infty$ in the Lebesgue sense. Such functions may have more violent discontinuities than those of L_2, but they have to approach zero at $\pm\infty$ somewhat faster. Thus,

$$(6.4) \qquad \frac{1}{x^{2/3}\sqrt{x^2+1}}, \quad e^{-x^4}, \quad \frac{\sin x}{x^2+1}, \text{etc.}$$

belong to L_1 and so have the property (d). However, functions of L_1 and L_2 do not have identical properties with regard to their Fourier transforms, particularly in regard to the way the definition (2.1) is to be interpreted. Moreover the transform of a function of L_2 is again a function of L_2, while the transforms of functions of L_1 need not belong to either class. The first function of (6.3) does not belong to L_1 because it goes to zero too slowly at $\pm\infty$; while the first function of (6.4) does not belong to L_2 because its discontinuity at zero is too violent; (it approaches ∞ too fast).

Returning again to formal considerations, we find that the property (d) is very useful because it enables us to calculate many new definite integrals. For instance, if we denote the transform of* $\exp(-|x|)$ by $F(x)$, we have

*The symbol $|x|$ means the absolute value (numerical value) of x, and thus $|x|=x$ when x is positive, and $|x|=-x$ when x is negative. It is never negative, and $|x|=|-x|$ for all values of x.

$$\sqrt{2\pi}F(x) = \int_{-\infty}^{\infty} e^{isx}e^{-|s|}ds = \int_{0}^{\infty} e^{isx}e^{-s}ds + \int_{-\infty}^{0} e^{isx}e^{s}ds$$

$$= \left[\frac{e^{s(ix-1)}}{ix-1}\right]_0^{\infty} + \left[\frac{e^{s}(ix+1)}{ix+1}\right]_{-\infty}^{0}$$

$$= -\frac{1}{ix-1} + \frac{1}{ix+1} = \frac{-2}{(ix-1)(ix+1)} = \frac{2}{x^2+1}\ ;$$

so

$$e^{-|x|} \rightleftarrows \sqrt{\frac{2}{\pi}} \cdot \frac{1}{x^2+1}\ .$$

Now applying rule (d), we reverse the order and have

$$\text{(6.5)} \qquad \sqrt{\frac{2}{\pi}} \quad \frac{1}{x^2+1} \rightleftarrows e^{-|-x|}\ ;$$

and this when written out gives us the new integration formula

$$\int_{-\infty}^{\infty} e^{isx} \quad \frac{ds}{s^2+1} = \pi e^{-|x|}\ .$$

The correctness of this formula may be checked by means of a table of definite integrals.

7. *List of the formal properties of Fourier transforms.* To facilitate further formal calculations, it seems worth while to collect the various properties of Fourier transforms into one list; and we include in this list both the formulas already obtained and those which will be obtained later:

If c is a real or complex constant, τ a real constant, and

$$f(x) \rightleftarrows F(x) \quad \text{and} \quad g(x) \rightleftarrows G(x),$$

then it follows that

(a)
$$f(x)+g(x) \rightleftarrows F(x)+G(x)$$
$$c\,f(x) \rightleftarrows c\,F(x)$$

(b)
$$ix\,f(x) \rightleftarrows \frac{d}{dx}\,F(x)$$

(c) $$\frac{d}{dx} f(x) \rightleftarrows -ix\,F(x)$$

(d) $$F(x) \rightleftarrows f(-x)$$

(e) If $F(x) = G(x)$, then $f(x) = g(x)$

(f) $$e^{i\tau x} f(x) \rightleftarrows F(x+\tau)$$

(g) $$f(x+\tau) \rightleftarrows e^{-i\tau x} F(x)$$

(h) $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-u) g(u)\,du \rightleftarrows F(x) G(x)$$

(i) $$f(x) g(x) \rightleftarrows \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x-u) G(u)\,du.$$

8. *Uniqueness.* From property (d) there follows immediately another important property of the Fourier transformation, namely:

(e) *It is a one-to-one transformation.* This means that to each one single function $f(x)$ there corresponds only one transform $F(x)$; and conversely that each one transform $F(x)$ is the transform of only one function $f(x)$. A function cannot have two transforms (as we see from the definition, which is not multiple valued); and a single transform cannot belong to two distinct functions. The latter fact is deeper, and says that if $F(x) = G(x)$, then $f(x) = g(x)$; or in terms of integrals, if

$$\int_{-\infty}^{\infty} e^{isx} f(s)\,ds = \int_{-\infty}^{\infty} e^{isx} g(s)\,ds \quad \text{for all real } x,$$

then
$$f(x) = g(x) \quad \text{for all* real } x.$$

This theorem holds for all classes of functions for which the property (d) holds, as the following argument shows. For if $f(x) \rightleftarrows F(x)$ and $g(x) \rightleftarrows G(x)$ then $F(x) \rightleftarrows f(-x)$ and $G(x) \rightleftarrows g(-x)$; so if $F(x) \equiv G(x)$, then $f(-x) \equiv g(-x)$, and $f(x)$ and $g(x)$ are identical.

This uniqueness property has actually been used before we formally stated it. Thus, in section 5 we drew the conclusion (5.4) by noting that the right numbers of (5.2) and (5.3) are equal and assuming

*Actually the conclusion is true for all x except a set of Lebesgue measure zero. Thus, if $f(x)$ and $g(x)$ were equal with the exception of one single value of x where they differed, the integrals would still be equal for all x. But sets of Lebesgue measure zero are negligible for all of the calculations in which we are interested, and we therefore consider $f(x)$ and $g(x)$ as being equivalent.

that that implied that the left members were equal. This amounts to assuming that (e) holds.

9. *Translation properties and difference equations.* Another type of equation that Fourier transforms help to solve is the type known as difference equations, in which different values of the independent variable occur in the same function. Thus,

$$f(x)+f(x+1)+f(x+2)=e^x$$

is called a difference equation; while (1.5) is called a difference-differential equation because the derivatives of $f(x)$ occurs as well as $f(x)$ and $f(x+1)$. The reason Fourier Transforms can be applied in certain difference equations is that the Fourier transformation has the following translation properties:

(f) *It replaces a multiplication by* $\exp(i\tau x)$ by a translation of τ units to the left, and

(g) *Similarly, it replaces a translation of* τ *units to the left by multiplication by* $\exp(-i\tau x)$.

If we write these statements out symbolically they read

$$e^{i\tau x}f(x) \rightleftarrows F(x+\tau)$$

$$f(x+\tau) \rightleftarrows e^{-i\tau x}F(x),$$

and in terms of integrals they read

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(x+\tau)}f(s)ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}[e^{i\tau s}f(s)]ds$$

and $$e^{-i\tau x}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}f(s)ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}f(s+\tau)ds.$$

Written in this form, the truth of the first statement is self-evident and the second is seen to be correct as soon as we put the factor $\exp(-i\tau x)$ under the integral sign in the first member and replace s by $s'-\tau$ in the second member. The statement (g) may also be obtained from (f) by the use of (d).

Now to solve (1.5), let $f(x) \rightleftarrows F(x)$, and transform the equation

$$\frac{d}{dx}f(x)+f(x)+f(x+1) = \frac{1}{x^2+1}$$

by applying the formulas listed in section 7. Thus we have

$$\frac{d}{dx}f(x) \rightrightarrows -ix\,F(x)$$

$$f(x+1) \rightrightarrows e^{-ix}F(x)$$

and $$\frac{d}{dx}f(x)+f(x)+f(x+1) \rightrightarrows -ixF(x)+F(x)+e^{-ix}F(x).$$

But by (6.5), $$\frac{1}{x^2+1} \rightrightarrows \sqrt{\frac{\pi}{2}}\,e^{-|x|} \quad ,$$

so the transformed equation is

$$(-ix+1+e^{-ix})F(x) = \sqrt{\frac{\pi}{2}}\,e^{-|x|} .$$

Solving for $F(x)$, we find

$$F(x) = \sqrt{\frac{\pi}{2}} \quad \frac{e^{-|x|}}{(-ix+1+e^{-ix})} ;$$

and since by (d)

$$F(x) \rightrightarrows f(-x) \quad ,$$

it follows that

$$f(-x) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{isx} \frac{e^{-|s|}}{(-is+1+e^{-is})}\,ds;$$

so

$$f(x) = \tfrac{1}{2}\int_0^{\infty} \frac{e^{-isx}e^{-s}ds}{1+e^{-is}-is} + \tfrac{1}{2}\int_{-\infty}^{0} \frac{e^{-isx}e^{s}ds}{1+e^{-is}-is} .$$

We next substitute $-s$ for s in the second integral and write the complex exponentials in terms of trigonometric functions:

$$f(x) = \tfrac{1}{2}\int_0^{\infty} \frac{(\cos sx - i\sin sx)e^{-s}ds}{1+\cos s - i\sin s - is}$$

$$+\tfrac{1}{2}\int_0^{\infty} \frac{(\cos sx + i\sin sx)e^{-s}ds}{1+\cos s + i\sin s + is} .$$

Finally, we put the two fractions together under one integral sign, adding the fractions in the usual way by first reducing to common

denominator. All the imaginary terms now drop out, and we obtain the final answer

$$f(x)=\int_0^\infty \frac{[\cos sx+\cos(sx-s)+s\sin sx]}{2+2\cos s+2s\sin s+s^2}e^{-s}ds\ .$$

Of course we need to check this answer; for we have obtained it by purely formal manipulations based on very shaky logical foundations; and we have not attempted to justify each step by seeing that the functions are of the type to which the formulas apply. However, questionable methods of arriving at an answer do not invalidate the correctness of an answer if it actually satisfies the required equation; and it is easy to verify that the integral we have just obtained does converge and does satisfy the required equation. We of course do not claim that this is the only solution.

10. *Certain integral equations.* The integral equations that Fourier transforms help us to solve are those in which the unknown function occurs in what is called a *convolution* or *faltung* or *resultant*. The convolution $h(x)$ of two functions $f(x)$ and $g(x)$ may* be defined to be

$$h(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-s)g(s)ds\ ;$$

so that a convolution is a doubly infinite integral of the product of the two functions, the variables $x-s$ and s being substituted in the functions in place of x. Of course any other letter would do in place of s, and it does not matter in which function we substitute the s. For if $x-s=t$, we have

$$\int_{-\infty}^{\infty} f(x-s)g(s)ds=\int_{-\infty}^{\infty} f(t)g(x-t)dt\ ;$$

since the change of sign in the differential nullifies the change of sign due to the necessary interchange of limits after substitution.

Integral equations involving convolutions frequently arise in physics and other branches of applied mathematics, and it is therefore important to know how to solve them. The reason that we can solve

*The constant $1/\sqrt{(2\pi)}$ is only included in the definition for convenience. The term *convolution* does not necessarily include this constant.

them is that the Fourier transformation changes convolutions into ordinary products. The transformation has properties (h) and (i):

(h) *It replaces convolutions by products.*

(i) *It replaces products by convolutions.* Written symbolically, these statements are

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-u)g(u)du \rightleftarrows F(x)G(x)$$

and

$$f(x)g(x) \rightleftarrows \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x-u)G(u)du\ ;$$

and written out in detail, they are

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\left[\int_{-\infty}^{\infty} f(s-u)g(u)du\right]ds$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{itx}f(t)dt\right]\cdot\left[\int_{-\infty}^{\infty} e^{iux}g(u)du\right]$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}f(s)g(s)ds$$

$$=\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\left\{\left[\int_{-\infty}^{\infty} e^{is(x-u)}f(s)ds\right]\cdot\left[\int_{-\infty}^{\infty} e^{itu}g(t)dt\right]\right\}du.$$

The first of these statements has a simple formal proof based on interchanging order of integration and replacing $s-u$ by t:

$$\int_{-\infty}^{\infty} e^{isx}\int_{-\infty}^{\infty} f(s-u)g(u)duds=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{isx}f(s-u)g(u)duds$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix(s-u)}f(s-u)e^{ixu}g(u)dsdu.$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixt}f(t)e^{ixu}g(u)dtdu$$

$$=\int_{-\infty}^{\infty} e^{ixu}g(u)\left[\int_{-\infty}^{\infty} e^{ixt}f(t)dt\right]du$$

$$=\left[\int_{-\infty}^{\infty} e^{ixt}f(t)dt\right]\cdot\left[\int_{-\infty}^{\infty} e^{ixu}g(u)du\right].$$

The second statement may be obtained by combining the first statement with (d).

It is now possible to solve the first problem mentioned in the introduction:

$$\int_{-\infty}^{\infty} f(x-t)f(t)dt+2f(x)=g(x) \ ;$$

for we let $\qquad f(x)\rightleftarrows F(x) \quad \text{and} \quad g(x)\rightleftarrows G(x),$

and have $\qquad \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x-t)f(t)dt\rightleftarrows F(x)\cdot F(x)$

so that $\qquad \displaystyle\int_{-\infty}^{\infty} f(x-t)f(t)dt+2f(x)\rightleftarrows\sqrt{2\pi}[F(x)]^2+2F(x)$

and $\qquad \sqrt{2\pi}[F(x)]^2+2F(x)=G(x).$

But this transformed equation is an ordinary quadratic equation in $F(x)$, and we can solve it by the quadratic formula, obtaining

$$F(x)=\frac{-2\pm\sqrt{4+4\sqrt{2\pi}\,G(x)}}{2\sqrt{2\pi}}$$

$$=\frac{1}{\sqrt{2\pi}}\left(-1\pm\sqrt{1+\sqrt{2\pi}\,G(x)}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left[-1\pm\sqrt{1+\int_{-\infty}^{\infty} e^{isx}g(s)ds}\right].$$

Transforming back by the inverse Fourier transformation (6.1), we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\left[-1\pm\sqrt{1+\int_{-\infty}^{\infty} e^{ist}g(s)ds}\right]dt.$$

Though this is formally two solutions, the formal statement is very deceptive in this regard. It may, as a matter of fact, represent infinitely many different solutions, because we may choose the signs one way for some values of t and the other way for other values of t. However, there is one important case in which there cannot be more than one solution. This is the case in which $g(x)$ is of class L_1 and its Fourier transform is nowhere equal to $-1/\sqrt{(2\pi)}$ and we are seeking for solutions $f(x)$ of class L_1. In this case it can be shown that there is not more than one solution of class L_1. Moreover there is a simple rule for determining whether there is one or no solution. This consists of tracing out the values taken on by the Fourier transform of $g(x)$ as x varies continuously from $-\infty$ to $+\infty$. These values will trace a continuous curve in the complex plane, beginning and ending at zero. If this curve winds an even number of times around the point $-1/\sqrt{(2\pi)}$, there will be a solution of class L_1; but if it winds an odd number of times around $-1/\sqrt{(2\pi)}$, there will be no solution of class L_1. The reason is roughly that the square root must assume the value $+1$ at both $-\infty$ and $+\infty$ if the outside integral is to converge; and this can only happen if the expression under the radical winds an even number of times around the origin. In particular, if

$$g(x)=\frac{4x^2+10}{\pi(x^4+5x^2+4)}\,,$$

we obtain

$$f(x)=\frac{1}{\pi(x^2+1)}$$

as the unique solution of class L_1.

11. *A linear integral equation.* While dealing with integral equations it seems worth while to take up the more ordinary case of the linear integral equation. We shall take such a case for our last illustrative example. Let us therefore consider the equation given in (1.4), namely

$$\rho(x)+\int_0^\infty \rho(x-t)e^{-t}dt=\frac{1}{x^2+1}\,.$$

The integral in this equation does not appear to be a convolution because it is only taken from 0 to $+\infty$ instead of from $-\infty$ to $+\infty$. However, we can replace the lower limit by $-\infty$ if we replace $\exp(-t)$

by a function $g(t)$ which equals $\exp(-t)$ whenever t is positive but equals zero when t is negative. For then we have

$$\int_{-\infty}^{\infty} \rho(x-t)g(t)dt = \int_{-\infty}^{0} \rho(x-t)g(t)dt + \int_{0}^{\infty} \rho(x-t)g(t)dt$$

$$= \int_{-\infty}^{0} \rho(x-t)\cdot 0\cdot dt + \int_{0}^{\infty} \rho(x-t)e^{-t}dt = \int_{0}^{\infty} \rho(x-t)e^{-t}dt\ ;$$

and the equation becomes

$$\rho(x)+\int_{-\infty}^{\infty} \rho(x-t)g(t)dt = \frac{1}{x^2+1}\ .$$

Now if $\rho(x) \rightleftarrows \Phi(x)$ and $g(x) \rightleftarrows G(x)$, it follows that

$$\rho(x)+\int_{-\infty}^{\infty} \rho(x-t)g(t)dt \rightleftarrows \Phi(x)+\sqrt{2\pi}\,\Phi(x)G(x)\ ;$$

and since

$$\frac{1}{x^2+1} \rightleftarrows \sqrt{\frac{\pi}{2}}\, e^{-|x|}\ ,$$

we obtain the transformed equation

$$\Phi(x)[\,1+\sqrt{2\pi}\,G(x)\,] = \sqrt{\frac{\pi}{2}}\, e^{-|x|}\ .$$

But $$G(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}g(s)ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{isx}\cdot o\cdot ds + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{isx}e^{-s}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{s(ix-1)}ds = \frac{1}{\sqrt{2\pi}}\ \frac{1}{1-ix}$$

and hence $$\Phi(x)\left[1+\frac{1}{1-ix}\right] = \sqrt{\frac{\pi}{2}}\, e^{-|x|}$$

and $$\Phi(x) = \sqrt{\frac{\pi}{2}}\left(\frac{1-ix}{2-ix}\right)e^{-|x|}\ .$$

Transforming back, we have

$$\rho(x)=\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-isx}\,\frac{(1-is)}{(2-is)}\,e^{-|s|}ds\ ;$$

and after the usual simplification, we obtain the answer given in the introduction. Substitution shows that this is correct.

12. *Integrals allied to the Fourier transform.* Let us consider a function $f(x)$ defined on the interval from 0 to $+\infty$; and from this function let us construct four functions, all of which are defined on the whole interval from $-\infty$ to $+\infty$, as follows:

$$f_c(x)=\begin{cases} f(x) & \text{when } x \text{ is positive} \\ f(-x) & \text{" " " negative}\end{cases}$$

$$f_s(x)=\begin{cases} -i\,f(x) & \text{" " " positive} \\ i\,f(-x) & \text{" " " negative}\end{cases}$$

$$f_l(x)=\begin{cases} \sqrt{2\pi}\,f(x) & \text{" " " positive} \\ 0 & \text{" " " negative}\end{cases}$$

$$f_m(x)=\sqrt{2\pi}\,f(e^x) \quad \text{for all real } x.$$

Then if $F_c(x)$, $F_s(x)$, $F_l(x)$, $F_m(x)$ are the Fourier transforms of $f_c(x)$, $f_s(x)$, $f_l(x)$, $f_m(x)$, we obtain by formal substitution in the definition (2.1) and formal simplification the integrals

$$F_c(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(s)\cos sx\,ds$$

$$F_s(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(s)\sin sx\,ds$$

$$F_l(ix)=\int_0^{\infty} f(s)e^{-sx}ds$$

$$F_m(ix)=\int_0^{\infty} f(t)t^{x-1}dt\ .$$

These four important integrals are known as the Fourier cosine transform, the Fourier sine transform, the Laplace transform, and the Mellin transform respectively of $f(x)$. They have properties somewhat similar to those of the Fourier transforms, yet differing from them in many important points. For a discussion of these properties, the reader should consult Titchmarsh's *Introduction to the Theory of Fourier Integrals.*

A different type of modification of the Fourier transform is obtained when we deal with functions of a complex variable and use some path of integration other than the real axis. Thus the modified Fourier transform is

$$F(x)=\frac{1}{\sqrt{2\pi}}\int_C e^{isx}f(s)ds \tag{12.1}$$

where the contour C is chosen so that as many as possible of the formal properties of Fourier transforms still hold. In particular, C should either be a closed curve or a curve which goes to infinity in some direction at both ends. This is necessary to preserve property (c); for if there are finite end points, the values at these end points will have to be substituted in the UV term of the integration by parts, and an extra term will crop up and spoil property (c). If the contour is infinite, it must of course be chosen so that $f(s)$ approaches zero as we approach infinity.

The contour integral (12.1) satisfies properties (a), (b), and (c); and since these were the only properties used in section 4 in solving (1.2), this integral could have been used there instead of the integral going from $-\infty$ to $+\infty$. But the only place in which the integral itself was used was in the last step, where we obtain $Y(x)$ from $y(x)$. Thus the only difference that the use of (12.1) could make would be that the final integral would be taken over C instead from $-\infty$ to $+\infty$. But such an answer would have just as much formal justification as the one we actually obtained, and this leads us to wonder whether every contour would produce the same answer or would at least produce a solution to the problem. As a matter of fact, every contour along which the integral converges properly does give us a solution to the equation; but not necessarily the same solution; and this enables us to complete the solution of the problem. You remember that we obtained only part of the general solution and mentioned that a method would later be given by which we could obtain the rest. That method is merely to obtain different parts of the solution by varying the contour used, and then to add these parts together to form

the general solution. Since the equation with which we are dealing is linear and has its right hand member zero, the sum of two solutions is again a solution, and this process is valid.

In the present case, our solution is to be a multiple of

$$\int_C e^{i(sx-(s^3/3))}e^{-(s^2/2)}ds,$$

where C is to be chosen so as to approach infinity in some other way than positively and negatively along the real axis. We wish the integrand to approach zero, and this means that the real part of the exponent

$$i(sx-(s^3/3))-(s^2/2)$$

must approach $-\infty$. Now for numerically large s, the numerically largest term is $-is^3/3$; and this will be real if s is pure imaginary. If $s=it$, then $-is^3/3=-t^3/3$, and it approaches $-\infty$ as s approaches infinity. Thus we can let one end of C go out along the upper part of the imaginary axis; and of course the other end can go out in either of the directions used before. Let us therefore take C as a contour starting at $i\infty$ and coming down the imaginary axis to zero, and then turning right and going out along the real axis to $+\infty$. Using this contour, we have

$$-Bi\int_c e^{i(sx-(s^3/3))}e^{-(s^2/2)}ds$$

$$=-Bi\int_\infty^0 e^{i(itx+(it^3/3))}e^{t^2/2}i\,dt-Bi\int_0^\infty e^{i(sx-(s^3/3))}e^{-(s^2/2)}ds$$

$$=-B\int_0^\infty e^{-(t^3/3)+(t^2/2)-tx}dt$$

$$-Bi\int_0^\infty [\cos(sx-(s^3/3))+i\sin(sx-(s^3/3))]e^{-(s^2/2)}ds.$$

We may as well drop the cosine term of this solution, as it is just like the part already obtained and is therefore itself a solution and may

be included with the first part by a change of the constant A. We therefore have as the second part of our solution

$$B\int_0^{\infty}\left[e^{-(s^2/2)}\sin(sx-(s^3/3))-e^{-(s^3/3)+(s^2/2)-sx}\right]ds;$$

and the problem is completely solved. The convergence of this second integral and the fact that it is a solution of the differential equation can be directly verified; and it can also be shown that neither part of the solution is identically zero or a constant multiple of the other part.

13. *Conclusion and warning.* One frequently hears the statement that a little knowledge of medicine is a dangerous thing. A similar statement might well be made in regard to certain branches of mathematics; particularly Fourier transforms. A mere formal knowledge of Fourier transforms will lead the manipulator to all sort of false conclusions. The situation here is much worse than in elementary calculus where lack of mathematical rigor and optimistically formal use of limit theorems *may* lead to false conclusions, but usually do not. Here purely formal work is sure to lead one into difficulties, and rather soon at that. You see, in this work there is really nothing that can be called "formally correct", because the formal rules are not even self consistent unless we put strong restrictions on the functions; and when we begin to state these restrictions there is no half way about it. We have to go all the way and do exact, rigorous mathematics.

As we stated in the introduction, our purpose here is merely to give the reader a general idea of the way a Fourier transform ordinarily behaves when suitably restricted. We hope that the reader may become interested in Fourier transforms as he sees the sort of thing that can be done with them, and that he may be willing to take time to learn the details of their exact behavior after he has had this little non-technical glance at the way they act when the machinery is well oiled with sufficiently powerful hypotheses. If the reader would like to gain a real understanding of the subject, he should study one of the standard works such as Titchmarsh's *Theory of Fourier Integrals,* Carslaw's *Fourier Series and Integrals,* Bochner's *Vorlesungen über Fouriersche Integrale,* Wiener's *The Fourier Integral,* and Paley and Wiener's *Fourier Transforms in the Complex Domain.*

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By BENJAMIN F. FINKEL
Drury College

(Continued from March, 1941, issue)

THE MATHEMATICAL MONTHLY

By J. D. RUNKLE, (*Continued*)

Believing that there may be many articles in this periodical that might prove of value to many readers, were the articles made known to them, we here reproduce the entire table of contents:

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*Professor Peirce's note reads as follows: The symbols which are now used to denote the Naperian base and ratio of the circumference of a circle to its diameter are, for many reasons, inconvenient; and the close relation between these two quantities ought to be indicated in their notation. I would propose the following characters, which I have used with success in my lectures: Here Professor Peirce produces his two symbols, the first to denote the ratio of the circumference to the diameter, and the other to denote the Naperian base. He says further, It will be seen that the former symbol is a modification of the letter *c* (circumference)—and the latter of *b* (base).

The connection of these quantities is shown by the equation $b^c=(-1)-\sqrt{-1}$, using the English symbols suggested in this sentence. In modern notation, this equation would be written

$$e^{\pi}=(-1)-\sqrt{-1}.$$

As Professor Peirce's symbols may be easily made, the writer will give a method of constructing them approximately after which the reader can construct them and substitute them in the last equation.

For symbols to be used with type the size used in this Magazine, invert a Capital U, begin at the midpoint between the extremities of the two parallel sides of the inverted "U" and draw a straight line parallel and equal to them; for π, connect the lower extremity of the middle line with the lower end of the right side of the inverted U with a concave semi-circle. To make the symbol for "e", the base of the Naperian system of logarithms, proceed as before, except, connect the lower extremity of the middle line to the extremity of the left parallel side of the inverted "U" with a concave semi-circle.

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With volume III, *The Mathematical Monthly* ceased publication, though nowhere in the volumes possessed by the writer, did the Editor give any notice of his intention to discontinue its publication. However, the Civil War having begun to manifest its disorganizing and destructive effect on the progress and development of the country may have had all to do with the final decision of its Editor.

In bringing this review of *The Mathematical Monthly* to a close, it is fitting that a few biographical remarks be made concerning its eminently successful and efficient Editor.

J(ohn) D(aniel) Runkle was born Oct. 11, 1822 and died July 8, 1902. He was born at Root, Montgomery County, N. Y., and spent his early life on a farm. Although handicapped at this time in his efforts to secure an education, he persevered, and at the age of twenty-five, he was able to enter the newly established Lawrence Scientific School at Harvard. In the Harvard Catalogue of 1848-1849, his name stands alone "Student in Mathematics". He was a member of the first graduating class of the Lawrence Scientific School, receiving the two degrees of S. B. and M. A., simultaneously, in 1851; after his graduation, he sent two of his four younger brothers to Harvard. In 1852, he contributed to the *Astronomical Journal* on the elements of Thetis and Psyche. In 1856, his *new tables for determining the values of the coefficients in the Perturbative Function of Planetary Motion*, were published in the Smithsonian Contributions to Knowledge and other tables followed in 1857. In 1858, as we have seen, he began the publication of *The Mathematical Monthly*. In 1849, he had begun a connection with the Nautical Almanac, a relationship which continued in one way or another for thirty-five years.

In 1862, Runkle married Catherine Robbins. In that same year, the Massachusetts Institute of Technology was founded and Runkle became its first secretary. He was Professor of Mathematics in the Institute from 1865 to 1868 and from 1880 to 1902 when he was made Professor Emeritus. In 1868, when illness incapacitated President William Barton Rogers, Runkle became acting president and from 1870 to 1878 he was president.

For a little fuller account of Mr. Runkle, consult the *Dictionary of American Biography,* from which the foregoing was taken.

Those submitting papers for publication in this journal are requested to read carefully instructions about manuscripts. Such instructions are to be found on title page.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Simplification by Rotation

By F. H. STEEN
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1. *Introduction.* The standard treatment of the topic *Simplification by Rotation* in analytic geometry involves the derivation of the rotation formulas and proof of the fact that the xy term is eliminated when the angle of rotation φ is chosen according to the relationship $\tan 2\varphi = B/(A-C)$. Then, in a numerical problem, $\sin\varphi$ and $\cos\varphi$ are found from this relationship and are placed into the rotation formulas. The values obtained for x and y in terms of x' and y' are substituted into the given equation and the result expanded and then simplified. The final equation and the angle φ are used to sketch the curve.

An algebraic treatment of the topic, as suggested in this paper, seems to have many advantages. The transformation equations are obtained at sight from a figure, the development is no more difficult than from the trigonometric point of view, the results are easily learned, and only the simplest numerical work is involved in an application.

2. *Development of Theory.* Let the perpendicular lines $y-mx=0$ and $x+my=0$ (m positive) be a new pair of axes and let the coordinates of a point $P(x,y)$ referred to these axes be x' and y', respectively (Fig. 1). Then, applying the formula for the distance from a line to a point in the usual way, with proper regard to choice of signs, we obtain

$$(1) \qquad x' = \frac{x+my}{\sqrt{1+m^2}}, \qquad y' = \frac{y-mx}{\sqrt{1+m^2}}.$$

Solving the linear equations (1) for x and y we obtain

$$(2) \qquad x = \frac{x'-my'}{\sqrt{1+m^2}}, \qquad y = \frac{y'+mx'}{\sqrt{1+m^2}}.$$

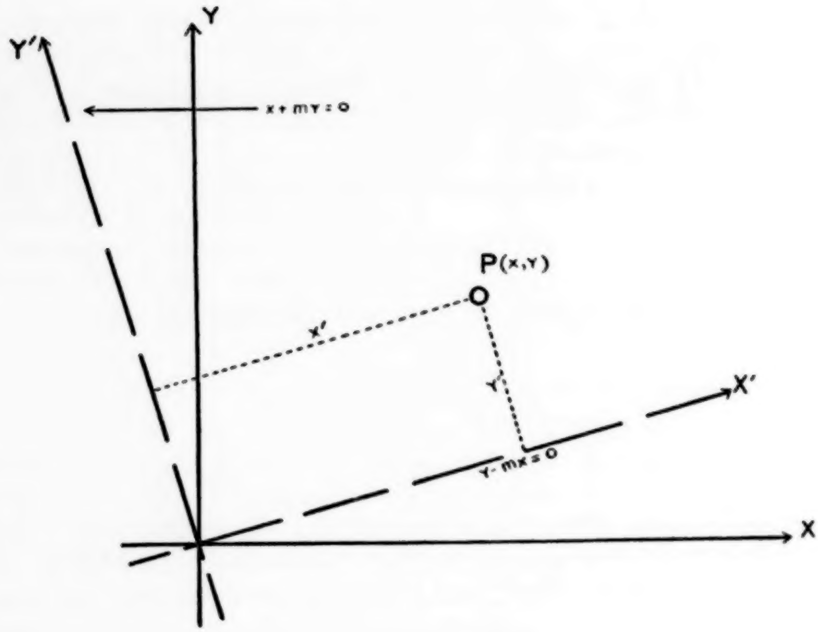

FIG. 1.

Substitution of these values of x and y in the general second degree equation

$$\text{(I)} \qquad Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

gives

$$\text{(II)} \qquad A'x'^2+B'x'y'+C'y'^2+D'x'+E'y'+F'=0.$$

The resulting coefficient B' of the $x'y'$ term is seen to be

$$\text{(3)} \qquad \frac{A(-2m)+B(1-m^2)+C(2m)}{1+m^2}$$

which is zero for

$$m=\frac{(C-A)\pm\sqrt{(C-A)^2+B^2}}{B}.$$

Setting $R=(C-A)/B$ and using the plus sign on the radical since m is positive, we have

$$\text{(4)} \qquad m=R+\sqrt{R^2+1}, \qquad R=(C-A)/B.$$

Knowing *m*, the slope of the x' axis, we can easily mark points S and T which when joined to the origin will give the x' and y' axes, respectively.

Let the (x,y) coordinates of S be (x_1,y_1). Then (see Fig. 2) the (x',y') coordinates of S are

$$x_1' = \sqrt{x_1^2+y_1^2}, \quad y_1' = 0.$$

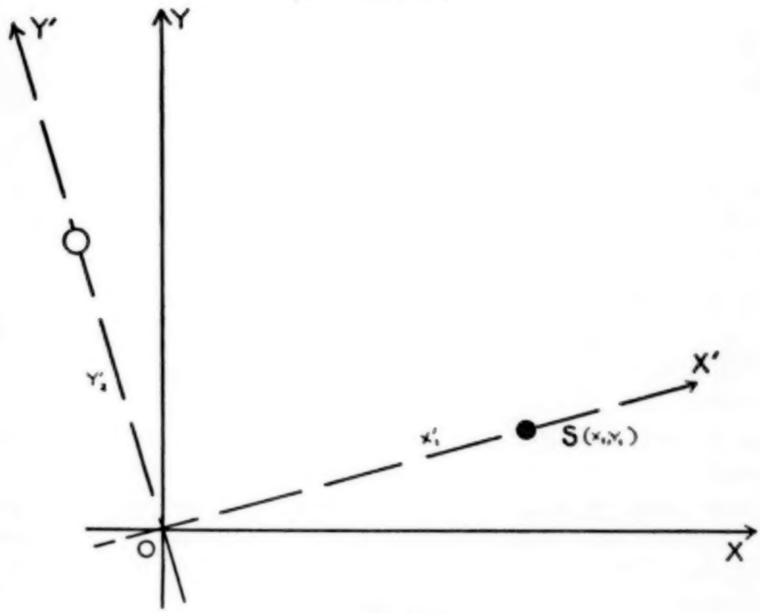

FIG. 2

If we substitute the coordinates of $S(x_1,y_1)$ into the quadratic terms of (I) we get the same result as when we substitute the coordinates of $S(x_1',y_1')$ into the quadratic terms of (II). This gives

$$Ax_1^2+Bx_1y_1+Cy_1^2 = A'(x_1^2+y_1^2)$$

or

(5) $$A' = \frac{Ax_1^2+Bx_1y_1+Cy_1^2}{x_1^2+y_1^2}.$$

The same substitution into the linear terms of (I) and (II) gives

$$Dx_1+Ey_1 = D'\sqrt{x_1^2+y_1^2}$$

or

(6) $$D' = \frac{Dx_1+Ey_1}{\sqrt{x_1^2+y_1^2}}.$$

Repeating the substitutions using a point $T(x_2,y_2)$ on the y' axis, we obtain

$$C'=\frac{Ax_2^2+Bx_2y_2+Cy_2^2}{x_2^2+y_2^2} \tag{7}$$

$$E'=\frac{Dx_2+Ey_2}{\sqrt{x_2^2+y_2^2}}\;. \tag{8}$$

Evidently

$$F'=F. \tag{9}$$

It follows from (6) and (8) that any radicals introduced by the rotation can be expressed rationally in terms of OS, x_1,y_1 and the original unit. This is important in sketching the curve.

SUMMARY: To remove the xy term from the equation (I) *of a conic, refer it to new axes* $y-mx=0$ *and* $x+my=0$, *choosing m as follows:*

$$m=R+\sqrt{R^2+1}, \qquad R=(C-A)/B.$$

The coefficients of x'^2 *and* x' *in the resulting equation* (II) *are obtained by substituting the coordinates of any point on* $y-mx=0$ *into the quadratic and linear terms, respectively, of* (I) *and dividing by the sum of the squares of the coordinates of this point and the square root of the sum of the squares, respectively, The coefficients of* y'^2 *and* y' *are obtained in a similar way using a point on* $x+my=0$. *The constant is unchanged.*

3. *Additional Relationships.* Some interesting and useful relationships among the constants of (I) and (II), true for all values of m, will now be derived.

Using $(-y_1,x_1)$ as (x_2,y_2) in (7) and adding to (5) we obtain

$$A'+C'=A+C. \tag{10}$$

Similarly, using equations (6) and (8),

$$D'^2+E'^2=D^2+E^2. \tag{11}$$

These relations serve as checks on computation. Equation (10) may be used for obtaining C' when A' has been found.

From (3) we obtain

$$(B'-B)=-2m(A+Bm-C)/(1+m^2)$$

$$(B'+B)=-2(Am-B-Cm)/(1+m^2).$$

Also, using $(x_1,y_1)=(1,m)$ in (5) we obtain

$$(A'-C)=(A+Bm-C)/(1+m^2)$$

$$(A-A')=m(Am-B-Cm)/(1+m^2)$$

Hence $\quad (B'-B)=-2m(A'-C)$

and $\quad (B'+B)=-2(A-A')/m.$

Multiplying the last two equations together and using (10) we obtain

(13) $$B^2-4AC=B'^2-4A'C'$$

which is another of the standard relationships.

4. *Applications.* Example 1. Simplify the following equation by a rotation: $5x^2-4xy+8y^2=36$.

Here $R=(8-5)/(-4)=-3/4$; $m=-3/4+\sqrt{(-3/4)^2+1}=1/2$.

Thus, the x' axis has a slope $1/2$ and the y' axis a slope -2. To draw these axes we join the points $S(2,1)$ and $T(-1,2)$ to the origin. Using the coordinates of these points in further applying the method we obtain

$$A'=(5\cdot 4-4\cdot 2\cdot 1+8\cdot 1)/(4+1)=20/5=4$$

$$C'=(5\cdot 1-4\cdot -1\cdot 2+8\cdot 4)/(1+4)=45/5=9.^*$$

and the new equation is

$$4x'^2+9y'^2=36.$$

Example 2. Simplify: $x^2+2xy+y^2+4x-4y=0$.

Here $\quad R=(1-1)/2=0;\quad m=0+\sqrt{0^2+1}=1.$

Using $S(1,1)$ and $T(-1,1)$ we obtain

$$A'=(1+2+1)/2=4/2=2;\qquad C'=(1-2+1)/2=0$$

$$D'=(4-4)/\sqrt{2}=0;\qquad E'=(-4-4)/\sqrt{2}=-4\sqrt{2}$$

and the new equation is

$$2x'^2-4\sqrt{2}\,y'=0,\quad \text{or}\quad x'^2=2\sqrt{2}\,y'.^\dagger$$

Example 3. Simplify: $4x^2-24xy+11y^2-24x+32y+40=0$.

Here $R=(11-4)/(-24)=-7/24$; $m=-7/24+\sqrt{(49/576)+1}=3/4$.

*Or, using (10), $C'=(5+8)-4=9$.

†Note that $OS=\sqrt{2}$.

Using $S(4,3)$ and $T(-3,4)$ we obtain

$$A'=-5,\quad C'=20,\quad D'=0,\quad E'=40,\quad \text{and}\quad F'=40$$

and the new equation is

$$-5x'^2+20y'^2+40y'+40=0$$

which can be written in the form

$$x'^2/4-(y'+1)^2/1=1.$$

Authors of papers being published in this journal are requested to retain a copy of the original manuscript so that it will be unnecessary for the Editor and Manager to forward a copy of the manuscript along with the galley proof to the author for correction of same.

Mathematical World News

Edited by
L. J. Adams

The complete program of the Mathematical Conferences at Stanford University, California on August 11, 12 is as follows:

1. *Truth and Mathematics*. Professor E. T. Bell, California Institute of Technology.
2. *Some Notes on Geometry of Numbers*. Professor H. F. Blichfeldt, Stanford University.
3. *Heuristic Reasoning and the Calculus of Probability*. Professor G. Polya, Brown University.
4. *Statistical Estimation by Means of Confidence Intervals and by the Bayes Theorem*. Professor J. Neyman, University of California, Berkeley.
5. *Independent Functions*. Professor A. Zygmund, Mount Holyoke College.
6. *Periodicity in Topology*. Professor G. T. Whyburn, University of Virginia.
7. *Functional Analysis* in Topological Group Spaces. Professor A. D. Michal, California Institute of Technology.
8. *On Differential Operators of Infinite Order*. Professor E. Hille, Yale University.
9. *Convex Domains and the Increase of Entire Functions*. Professor G. Polya, Brown University.
10. *Problems of the Theory of Trigonometric Series*. Professor A. Zygmund, Mount Holyoke College.

In addition there will be an informal reception on Monday, August 11, at 8:30 p. m.

The annual meeting of the National Academy of Sciences will be held at Washington, D. C., on April 28-30, 1941.

Mathematical Reviews, the international journal which contains abstracts and reviews of current mathematical literature, now appears monthly.

The British Association for the Advancement of Science proposes a "democratic charter of science" in collaboration with American scientists, to be used by scientists throughout the world. Sir Richard Gregory has submitted a preliminary charter, to be considered by a committee including Sir Richard Gregory, H. G. Wells, Professor Alan Ferguson, Ritchie Calder and Professor Hyman Levy.

Professor Frederick G. Reynolds, chairman of the mathematics department of the College of the City of New York, has resigned from the position of secretary of the faculty. In June, 1941 he will retire from the chair of mathematics.

The Southern California Section of the Mathematical Association of America met at the University of Redlands in Redlands, California on March 8, 1941. Professor O. Albert of the University of Redlands presided as chairman. The program follows:

1. *The place of mathematics in the curriculum.* Professor G. R. Livingston, San Diego State College.
2. *Derivatives and the study of curves.* Professor A. E. Taylor, University of California at Los Angeles.
3. *Mathematics in the Naval Reserve Training School.* Captain R. M. Fawell, University of Southern California.
4. *Methods for solutions of secular equations.* Professor J. H. Wayland, University of Redlands.
5. *Points of contact between mathematics and the civilian pilot training program.* Professor W. T. Whitney, Pomona College.
6. *The resistance of ships.* Professor Harry Bateman, California Institute of Technology.

Officers elected for the academic year 1941-42 are: Chairman, Mr. L. J. Adams, Santa Monica Junior College; Vice-Chairman, Professor Morgan Ward, California Institute of Technology; Secretary, Professor Paul H. Daus, University of California at Los Angeles. The program committee for the coming year consists of: Chairman, Dr. D. C. Duncan, Los Angeles City College; Professor C. K. Alexander, Occidental College; Professor Paul H. Daus, University of California at Los Angeles. The next meeting of the Section will be held on March 14, 1942 at Occidental College.

The American Mathematical Monthly for February, 1941 announces the publication of a new series of Papers named in honor of the late Professor Herbert Ellsworth Slaught, to be published by the Mathematical Association of America.

The following meetings of sections of the Mathematical Association of America are scheduled for the month of May:

1. Allegheny Mountain. Pittsburgh, May 3.
2. Indiana. Indianapolis, May 2-3.
3. Maryland-District of Columbia. Annapolis, Md., May.
4. Nebraska. Lincoln, May.
5. Upper New York State. Ithaca, May 3.
6. Wisconsin. Beloit, May 3.

The summer meeting of the Mathematical Association of America will be held at the University of Chicago, September 1-3, 1941. The annual meeting will be held at Bethlehem, Pennsylvania, on December 29, 1941-January 2, 1942.

Dr. D. T. McClay has been appointed instructor in mathematics at the Georgia School of Technology.

The Mathematics Section of the Southern California Junior College Conference is scheduled to meet at Chaffey Junior College in Ontario, California on April 19, 1941. The program follows:

1. *Admission Examinations for the California Institute of Technology.* Professor L. E. Wear, California Institute of Technology.
2. *Mathematical training for C. A. A. trainees.* Mr. W. W. Weber, Chief Instructor of Ground School, Cal-Aero Academy.
3. *Klein's Famous Problems in Geometry.* Dr. D. C. Duncan, Los Angeles City College.

An Engineering Defense Training Institute opened at Brooklyn Law School on February 10, 1941. There are some four hundred students, chosen from ten thousand applicants. Several engineering colleges are cooperating by furnishing their facilities to this Institute.

Profess G. A. Miller, University of Illinois, discusses *General or Special in the Development of Mathematics*, in the March 7, 1941 issue of *Science*.

On February 21, 1941 the American Mathematical Society held a symposium on Applied Mathematics at Columbia University. Dr. W. A. Shewart, Bell Telephone Laboratories, discussed *Mathematical Statistics in Mass Production.* During the course of his remarks he stated that statistical control can provide "a technique for modifyfying and coordinating the three fundamental steps in the process of mass production, namely, specification, manufacturing and inspection, so that the maximum number of pieces of product having a quality within specified tolerance limits can be turned out at given cast."

Dr. A. N. Whitehead, professor of philosophy emeritus of Harvard University, celebrated his eightieth birthday on February 15, 1941.

Dr. Lancelot Hogben, guest professor of genetics at the University of Wisconsin and widely known as the author of *Mathematics for the Million*, has returned to England.

Most of the articles in the March, 1941 issue of *The Mathematics Teacher* deal with the training of teachers of secondary mathematics.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

No. 377. Proposed by *V. Thébault*, San Sebastián, Spain.

Form two perfect squares whose sum shall be 148392.

Solution by *Edwin Comfort*, University of Arkansas.

$(2n)^2 \equiv 0$ mod 4, $(2n+1)^2 \cancel{\equiv 0}$ mod 4, $(3n)^2 \equiv 0$ mod 3, $(3n \pm 1)^2 \equiv 1$ mod 3 are obvious for every integer n. Put

$$x^2 + y^2 = 148392. \tag{1}$$

$148392 \equiv 0$ mod 4 implies that both x and y are even. With $x = 2a$, $y = 2b$, (1) becomes

$$a^2 + b^2 = 37098.$$

Since $37098 \equiv 0$ mod 3, both a and b are multiples of 3 and we put $a = 3c$, $b = 3d$ to find

$$c^2 + d^2 = 4122.$$

Similarly, $c = 3e$, $d = 3f$ gives

$$e^2 + f^2 = 458.$$

in which e and f must be odd. Substitution of $e = 2g+1$, $f = 2h+1$ (hence $x = 36g+18$, $y = 36h+18$) gives

$$\tfrac{1}{2}g(g+1) + \tfrac{1}{2}h(h+1) = 57.$$

Thus we require two triangular numbers with sum 57. From the first eleven triangular numbers, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, and 66,

the unique solution is seen to be $36+21=57$. Thus g and h are the numbers 6 and 8, whence x and y are 234 and 306.

Also solved by *H. T. R. Aude, Annie M. H. Christensen, W. B. Clarke, C. S. Cushman, D. L. MacKay, C. W. Trigg,* and the *Proposer.*

No. 378. Proposed by *Robert C. Yates*, Louisiana State University.

An ellipse moves so that it is always tangent to two perpendicular lines. Find the locus of a focus.

Solution by *H. T. R. Aude*, Colgate University.

Two well known properties of the ellipse are: (1) The feet of the perpendiculars from a focus to the tangents of an ellipse lie on the circle over the major axis of the ellipse as a diameter. (2) Perpendicular tangents to an ellipse intersect on the circle concentric with the ellipse. The radius of the circle is equal to the square root of the sum of the squares of the semiaxes of the ellipse.

With these two properties in mind choose the perpendicular lines for coordinate axes. Let the given ellipse have semiaxes a and b, $a>b$, and denote the coordinates of the center (α,β). Assume the ellipse in any position tangent to these axes. On account of the second property, it is true that

$$\alpha^2+\beta^2=a^2+b^2. \tag{1}$$

The circle on the major axis of the ellipse as diameter will have the equation

$$(x-\alpha)^2+(y-\beta)^2=a^2. \tag{2}$$

The intercepts of this circle on the x-axis and the y-axis are given respectively by the relations

$$(x-\alpha)^2=a^2-\beta^2, \quad (y-\beta)^2=a^2-\alpha^2. \tag{3}$$

On account of the first property the coordinates of a focus $P(x, y)$ are given by a set of the values of the variables x and y in (3). From the relations in (3) and (1) it follows that

$$2x\alpha=x^2+b^2 \quad \text{and} \quad 2y\beta=y^2+b^2. \tag{4}$$

Eliminating α and β between statements (4) and (1) gives the relation

$$[(x^2+b^2)/x]^2+[(y^2+b^2)/y]^2=4a^2+4b^2,$$

and this is the equation of the locus.

Also solved by *Charles H. Cunkle.*

No. 379. Proposed by *E. C. Kennedy*, Texas College of Arts and Industries.

Consider

$$T_n=\sqrt{\frac{k^2+T_{n-1}}{4-T_{n-1}}}, \qquad 0<T_0\leqq k/2.$$

What is the largest value of k^2 such that the sequence $\{T_n\}$ converges to a real positive number? What is the number?

Solution by the *Proposer*.

Assuming that $\{T_n\}$ has a positive limit T, we have at once

$$T=(k^2+T)^{\frac{1}{2}}/(4-T)^{\frac{1}{2}} \quad \text{or}$$

(1) $$T^3-4T^2+T+k^2=0.$$

This cubic has two positive roots or none according as its discriminant satisfies $\Delta\geqq 0$ or $\Delta<0$. Since $\Delta=12+184k^2-27k^4$, we have

$$k^2\leqq(92+26\sqrt{13})/27;$$

whence the desired maximum of k^2 is 6.8794, corresponding to $\Delta=0$.

Conversely, for certain values of k^2 the sequence $\{T_n\}$ converges to the smaller positive root of (1). See Problem No. 264, this Magazine for April, 1939, p. 348; also Kennedy, *Root Isolation Through Curve Analysis*, this Magazine for April, 1940, pp. 373-378. In the case $\Delta=0$, (1) has a positive double root, easily calculated, whence $T=(4+\sqrt{13})/3=2.4389$.

No. 380. Proposed by *N. A. Court*, University of Oklahoma.

The polar lines of a fixed line with respect to the spheres of a coaxal pencil lie on a quadric surface.

Solution by *Paul D. Thomas*, Norman, Oklahoma.

The problem is that of finding the envelope of the polar planes of a point with respect to the spheres of a coaxal pencil as the point describes a fixed straight line. Let the fixed line be $x=ar$, $y=br$, $z=c$. The polar planes of any point of this line with respect to the spheres:

$$x^2+y^2+z^2+2gx+d=0$$

with parameter g are

$$f\equiv arx+bry+cz+g(x+ar)+d=0.$$

For the envelope: $$f_g\equiv x+ar=0.$$

Eliminating r between f and f_g we have the equation of the quadric

$$ax^2+bxy+acz-ad=0.$$

Interesting special cases are noted for a, b, c zero.

Also solved by the *Proposer.*

No. 381. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

The locus of the point Q is a circle if Q is the foot of the perpendicular from a point P upon the polar of P with respect to the conic $x^2/t+y^2/b^2=1$, where t is a parameter.

Solution by *W. T. Short*, Oklahoma Baptist University.

The polar of $P:(x_1,y_1)$ with respect to the conic is

$$b^2x_1x+t(y_1y-b^2)=0.$$

The perpendicular to this through P is

$$ty_1(x-x_1)=b^2x_1(y-y_1).$$

Eliminating t between these equations we have as the locus of Q:

$$y_1(x^2+y^2)-x_1y_1x-(b^2+y_1^2)y+b^2y_1=0.$$

the equation of a circle.

Also solved by *Charles S. Cushman* and the *Proposer.*

No. 383.* Proposed by *E. P. Starke*, Rutgers University.

Find the number of triangles of all kinds whose sides are positive integers and whose largest side does not exceed a given number, K.

Solution by *George A. Yanosik*, New York University.

The number of triangles with at least one side equal to K, and with no side greater than K, is easily seen to be

$$K+(K-2)+\cdots+2=K(K+2)/4=n(n+1),\quad K=2_n \quad \text{or}$$
$$K+(K-2)+\cdots+1=(K+1)^2/4=m^2,\quad K=2_{m-1},$$

according as K is even or odd. Then the number of triangles whose greatest side is K or less is given by

$$\sum_{n=1}^{\frac{1}{2}K} n(n+1)+\sum_{m=1}^{\frac{1}{2}K} m^2=K(K+2)(2K+5)/24 \quad \text{or}$$

*Through an error, this problem also appears as No. 390.

$$\sum_{n=1}^{1/2(K-1)} n(n+1) + \sum_{m=1}^{1/2(K+1)} m^2 = (K+1)(K+3)(2K+1)/24,$$

according as K is even or odd.

Also solved by *W. Raymond Crosier*, and *G. W. Wishard*.

Editor's Note: Several of our contributors submitted solutions which failed to take account of those triangles whose greatest side is less than K.

No. 384. Proposed by *W. L. Roberts*, Colgate University.

In an attempt to locate an enemy cannon during the World War, microphones were stationed on a straight line at A, at B, 2,200 feet from A, and at C, 4,400 feet from A. The explosion reached B $1/2$ second and C $3/2$ seconds after it was heard at A. Assuming the speed of sound to be 1,100 feet per second, find how far the enemy cannon was from points A, B, and C and at what angles with the line ABC cannon must be adjusted at A, B, and C to destroy the enemy gun.

Solution by *D. L. MacKay*, Evander Childs High School, New York.

If G is the position of cannon, let $AG=x$. Then $BG=x+550$, $CG=x+1650$. Using the median formula,* we have:

$$x^2+(x+1650)^2=2(2200)^2+2(x+550)^2,$$

whence $\quad x=6875=AG;\quad BG=7425;\quad CG=8525.$

Triangle BGC is similar to the triangle with sides (31,27,8) whose inradius $r=10/\sqrt{(11)}$ and $\tan(C/2)=r/(s-c)=5/3\sqrt{(11)}=.5025$. Thus $\angle C=53.36°$. In like fashion we find angles A and GBA.

Also solved by *Walter B. Clarke* and *H. M. Zerbe*.

PROPOSALS

No. 407. Proposed by *Mathematics Club*, Tulane University.

Construct a square such that an interior point is distant 3, 4, and 5 units from three of its vertices.

No. 408. Proposed by *Howard D. Grossman*, New York City.

Prove $$1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\cdots=\frac{\log 2}{3}+\frac{\pi\sqrt{3}}{9}.$$

*The sum of the squares of two sides of a triangle equals half the square of the third side plus twice the square of the median to that side.—ED.

No. 409. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Construct a triangle given the lengths of the median and internal bisector issued from the same vertex, and the angle between them.

No. 410. Proposed by *V. Thébault*, San Sebastián, Spain.

In what system of numeration, with base less than 100, is the three-digit number 333 a perfect square?

No. 411. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given the angles B and E. Draw a line $ACDF$, parallel to a given line and cutting the sides of angle B in A and C, and those of angle E in D and F so that $AC=DF$.

No. 412. Proposed by *S. B. Townes*, University of Oklahoma.

If p is a prime of the form $4n+1$ and K is any positive integer, show:

(a) There exists a representation $x^2+y^2+pz^2$ of Kp if and only if K cannot be written in the form $4^r(8s+7)$, $r\geqq 0$, $s\geqq 0$.

(b) There exists a representation $x^2+y^2+2pz^2$ of Kp if and only if K cannot be written in the form $2\cdot 4^r(8s+7)$, $r\geqq 0$, $s\geqq 0$.

No. 413. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

A circle is described on the line joining the foci of the hyperbola $x^2/a^2-y^2/b^2=1$ as a diameter. Show that the locus of the pole of a variable chord of the hyperbola which is tangent to the circle is also the envelope of the chords of contact of pairs of orthogonal tangents drawn to the ellipse $x^2/a^2+y^2/b^2=1$.

No. 414. Proposed by *E. P. Starke*, Rutgers University.

Let α and β be the roots of $t^2-at-1=0$ and let $f(x)=x/(1-ax-x^2)$. Show that $f^{(n)}(0)=n!(\alpha^n-\beta^n)/(\alpha-\beta)$.

Bibliography and Reviews

Edited by
H. A. Simmons and John W. Cell

Advances and Applications of Mathematical Biology. By Rashevsky Nicolas. University of Chicago Press, Chicago, Ill., 1940. xiii+214 pages. $2.00.

"Mathematical Biology" is here to be taken as synonymous with the author's coinage "Mathematical Biophysics", and it denotes the elaborations of quantitative biological theories from premises which describe the properties of and relations among the physical elements, either as observed or as hypothecated. It is therefore broader in scope than the "*Biologie Mathematique*" of Kostitzin. But since this is a record of the personal research of the author and his associates, no reference will be found to Kostitzin, Volterra, Lotka, *et al.*, nor is any discussion of the field of genetics included. The book here under review follows by less than two years the same author's *Mathematical Biophysics* (University of Chicago Press, 1938). During this interval radical approximations have been introduced, and as a result, the mathematics in the second book is much more elementary than that in the first, and a considerably greater number of direct contacts have been made with the laboratory. Discussion of these contacts, however, is not a part of this review.

It is convenient to follow the author's earlier division, and to classify the problems treated according as they relate to the vegetative cell, to reacting (nerve) cells, or to the nervous system as a whole. Treatment of the cell, whether vegetative, or reacting, simply extends certain chapters of classical physics, whereas for the discussion of the nervous system postulates of a somewhat different order are required. In all, the first six chapters are devoted to the vegetative cell, the seventh to excitation in nerve, the eighth is a transition chapter, while the last four chapters deal with the nervous system.

The vegetative cell is studied as a seat of metabolic reactions enclosed by a permeable or semi-permeable membrane, and immersed in an infinite liquid medium from which it draws its metabolites. As such it gives rise to concentration gradients in its own interior as well as in the surrounding medium. It is, therefore, acted upon by the forces resulting from these gradients, and its surface is the seat of surface energy due to the surface tension. The possible effects of electrolytes and of differential permeability of the membrane to the dissociated ions are not considered.

An exact solution of the diffusion equations is, at best, not feasible even for a cell whose form does not vary, because of the diversity and complexity of the boundaries of different cells. Since, nevertheless, the behavior (growth and division) of the cells seems to be far more uniform than the cell shapes, the author pictures the cell as being essentially a cylinder or a spheroid, according to convenience. Moreover, in order to avoid partial differential equations entirely, he pictures the diffusion flow as being purely radial and purely axial, with both internal and external gradients uniform. Thus instead of speaking of the concentrations at a point, the author speaks of the "average" internal concentration, the "average" internal concentrations at the "ends" and that at the "sides", the "average" external concentration at the "ends" and that at the "sides", and the concentration at infinity. In this way only linear algebraic, and ordinary differential equations arise.

First, the differential equation in the average internal concentration of a single metabolite is derived in terms of the cellular dimensions, the reaction rate, the diffusion

coefficients, and the permeability. If the reaction rate is assumed constant this equation is linear. Next, while involving the chemical law of mass action, and postulating certain specific chemical reactions, it is assumed that the concentration equation holds for each of the metabolites. From these assumptions the steady-state reaction rates are determined.

The forces from diffusion and surface tension then come into consideration as possible causes of cell division. Diffusion forces are thought of as acting not only on the membrane, but also on colloidal particles within the cell. After an estimate is made as to the magnitude of the forces on these particles, Betti's theorem giving the rate of deformation of an elastic solid is employed along with the laws of plastic flow to determine the rate of elongation of the cell. For a nearly-spherical cell, an ordinary differential equation in the axial radius is obtained, from which it is possible to obtain a "critical" radius, beyond which a spherical cell will begin to elongate (and perhaps to divide), below which it will not. For a cell already in the process of division there is an ordinary differential equation in the radius of the "neck".

Chapter IV discusses two possible growth mechanisms. The first assumes the cause to be the accumulation of metabolic products and, certain special assumptions being made as to the nature of the reactions, there results an asymptotic limit to the size of the cell in the cases when the formal limit does not exceed the critical size. The other assumes the expansion of the cell membrane resulting from osmotic (or other) forces to be the cause of growth. Chapter V takes up the simultaneous variation of shape and internal concentration of a single metabolite, and considers qualitatively the equilibria which may arise. The spherical shape is always an equilibrium shape, as one must expect, but there may be two others when the physical constants are suitably related. Discussion of the vegetative cell is concluded with a brief consideration of the possibility of protoplasmic streaming as a result of an asymmetric placement of stroma and catalyzers (nucleus) within the cell.

In nerve, three phenomena require explanation and description: excitation, conduction, and transmission. The theory of the first is essentially formal. Stimulation is supposed to lead to the development, within the nerve fiber, of two antagonistic substances or states, the rate of development being some increasing function of the intensity of the stimulus, and the rate of dissipation being proportional to its own measure. Excitation is to occur when the ratio of the measure of the "excitatory" state to that of the "inhibitory" state exceeds a certain fixed amount called the "threshold". Thus the excitatory process is described by a pair of simultaneous linear differential equations of the first order. Conduction is passed by in the present work with a brief mention and reference to previous work.

The discussion of transmission provides the transition to theories of the nervous system. The same excitation equations are regarded as describing a process at the synapse (junction between terminus and origin of two nerve fibers); the use of such equations, which imply continuous activity, for describing the results of a rapid succession of discrete bursts of activity (action spikes) is justified as a kind of averaging process. At any synapse the difference between the measures of the two states or substances which results from the activity of the incoming fiber is taken as the measure of the intensity of the stimulus which acts upon the outgoing nerve fiber.

The topological relations among the nerve fibers which connect the receptor with the effector organs in a biological organism, together with the values of the parameters which are associated with each of the component fibers, and which occur in the equations for describing the manner in which the fiber develops the two antagonistic states, these are taken as the essential determiners of the reaction of the organism when it is placed in any given stimulus situation. All the special theories relating to the nervous system

are built up by postulating a particular neural structure and investigating the implicated activity (time-course of excitation in the constituent nerve fibers) as a function of the intensity of the peripheral stimulus. Thus for obtaining the time-interval between stimulus and reaction as a function of the stimulus intensity, a simple chain of nerve fibers is postulated; for predicting the effect of a warning stimulus, two parallel incoming fibers are taken as leading to a single outgoing fiber; parallel chains with mutually inhibitory connections, together with a postulated random influx of excitation independent of the stimuli, yield predicted distributions of psychophysical judgments (comparison of simultaneously presented stimuli); another complex of parallel fibers with mutually inhibitory interconnections yields the discrimable difference as a function of the stimulus intensity (comparison of successively presented stimuli); and the beginning of an æsthetic theory of visual forms is based on a network of fibers connecting cerebral centers governing the eye-muscles and a postulated "pleasure center". Where the steady state is supposed to have been reached with a constant stimulus, and the number of fibers supposed to be small, the equations are all finite and are either algebraic or involve exponentials. However, when many fibers are supposed to be acting, statistical smoothing leads to integral equations where the limits of integration must be so adjusted that the solution of the equation is nowhere negative. These equations, therefore, differ somewhat from the classical Volterra and Fredholm forms.

From a mathematical point of view, this book is far more interesting for the problems it raises than for those which it solves. To mention but a few of these, there are the partial differential equations of diffusion, whether with fixed or with moving boundaries, which the author side-steps at the outset, and later there are the equations which describe the forces acting on the cell. The author's ordinary differential equations of the concentrations of the metabolites have not been worked out for the non-stationary state even for a special postulated chain of chemical reactions. The general type of integral equations occurring in the theory of the nervous system provide interesting features. And a system of "Kirchoff equations" for describing nervous transmission is needed. If the mathematicians of the future are to derive their inspiration from biology, as they have done in the past from physics, they should find here a good beginning.

University of Chicago. ALSTON S. HOUSEHOLDER.

Elements of Calculus. By Abraham Cohen, D. C. Heath and Company, N. Y., 1940. v+583 pages; Answers, 35 pages. $3.50.

The author has designed this book for students in pure mathematics and for those intending to use the calculus in the sciences. The book contains more material than a class can cover in the limited time ordinarily scheduled for the first course in the calculus, for, as the author says in his preface, "it is more satisfactory to omit, if necessary, certain parts of a comprehensive text than to have to supply any missing material from outside sources."

Functional notation, graphs, and limits are briefly treated in Chapter 1. In the Appendix is a fuller exposition of the concept of a limit and the elementary theorems on limits with proofs. Differentiation formulas and exercises in their use are presented in two parts: Chapter 2, on algebraic functions, and Chapter 6, on transcendental functions.

The geometrical application of derivative as slope is made in Chapter 2. The applications to maxima and minima and to curve sketching are in Chapters 3 and 4.

The derivative as rate of change is deferred to Chapter 5, on physical applications. Further geometrical applications including curvature are made in Chapter 7 for curves in rectangular and polar coordinates. Most of the curves in this chapter are treated as special curves and are accompanied by names and figures.

The differential is delayed to Chapter 8 where it is defined and treated in relation to the increment of a function and applied as the approximate value of the increment. The brief treatment of differential is preparatory to the definition of integration as the process inverse to differentiation. Finding the integral of $u^m du$ is practiced without and with the introduction of a new variable. The few simple geometrical and physical applications in this chapter do not include the finding of an area under a curve.

The common methods of integration are fully dealt with in Chapter 9 and Chapter 10 so as to develop mastery of integration by means of a score of formulas. It is left to the instructor to lay out exercises in the use of a table of integrals. The definite integral as the limit of a sum and the common properties of definite integrals are treated in Chapter 11, where applications are made to finding the area under a curve and to the length of an arc. Improper integrals, Wallis' formulas, and Simpson's rules are included in this chapter.

Additional applications of the definite integral are presented in two chapters: in Chapter 12, areas and lengths of arcs in rectangular and polar coordinates, volumes by parallel sections and by hollow cylinders, surfaces of revolution, work done by a force, liquid pressure, and mean value of ordinates; in Chapter 13, mass, center of gravity, moment of inertia, and attraction.

The exposition of partial derivatives in Chapter 14 includes maxima and minima of functions of several variables, Euler's theorem on a homogeneous function, and exact differential. Chapter 15 on envelopes is brief and yet adequate for a first course.

In Chapter 16, on multiple integrals, the double integral is sufficiently treated with applications, but the triple integral is very briefly treated.

The matter on infinite series in Chapter 17 includes the Cauchy integral test, the comparison and ratio tests, and power series as preparatory to the concise exposition of Maclaurin's and Taylor's series.

Under *miscellaneous theorems and applications* in Chapter 18 there are gathered together Rolle's theorem, mean value theorem for differentiation, Taylor's theorem with forms of the remainder, and several common indeterminate forms.

In the Appendix, of sixty-nine pages, appear the further treatment of limits, already mentioned, the hyperbolic functions, the elements of solid analytic geometry, some formulas and theorems from algebra, mensuration, trigonometry, analytic geometry, and a table of 201 integrals.

Although there are about eighty more pages devoted to the differential calculus than to the integral, the book is fairly evenly balanced. There are 190 figures. The answers to the exercises are in a separate booklet.

Even though the author has made good use of much of the material of his earlier book, he has not made a mere revision: he has written more simply and put in more illustrative exercises and many more problems. The publishers have produced an attractive book by their excellent printing.

Vanderbilt University. WILSON L. MISER.